

Monotone waves for non-monotone and non-local monostable reaction-diffusion equations

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Abstract

We propose a criterion for the existence of monotone wavefronts in non-monotone and non-local monostable diffusive equations of the Mackey-Glass type. This extends recent results by Gomez *et al.* [19] proved for the particular case of equations with local delayed reaction. In addition, we demonstrate the uniqueness (up to a translation) of obtained monotone wavefront within the class of all monotone wavefronts (such a kind of conditional uniqueness was recently established for the non-local KPP-Fisher equation by Fang and Zhao). Moreover, we show that if delayed reaction is local then this uniqueness actually holds within the class of all wavefronts and therefore the minimal fronts under consideration (either pulled or pushed) should be monotone. Similarly to the case of the KPP-Fisher equations, our approach is based on the construction of an appropriate fundamental solution for associated boundary value problem for linear integral-differential equation.

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1. Introduction and main results

Introduction. In this work, we study the existence and uniqueness of *monotone* wavefronts $u(x, t) = \phi(x + ct)$ for the monostable delayed non-local reaction-diffusion equation

$$u_t(t, x) = u_{xx}(t, x) - u(t, x) + \int_{\mathbb{R}} K(x - y)g(u(t - h, y))dy, \quad u \geq 0, \quad (1)$$

when the reaction term $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ neither is *monotone* nor defines a *quasi-monotone* functional in the sense of Wu-Zou [47] or Martin-Smith [32] and when the non-negative kernel $K(s)$ is Lebesgue integrable on \mathbb{R} . Equation (1) is an important object of studies in the population dynamics, see [6, 14, 17, 20, 28, 29, 33, 34, 41, 42, 45, 48, 49, 50]. Taking

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formally $K(s) = \delta(s)$, the Dirac delta function, we obtain the diffusive Mackey-Glass type equation

$$u_t(t, x) = u_{xx}(t, x) - u(t, x) + g(u(t - h, x)), \quad u \geq 0, \quad (2)$$

another popular focus of investigation, see [19, 24, 44] for more details and references.

In the classical case, when $h = 0$, all wavefronts to the monostable equation (2) are monotone and, given a fixed admissible wave velocity c , all of them are generated by a unique front by means of translations. The same monotonicity-uniqueness principle is valid for certain subclasses of equations (2) with $h > 0$ (e.g. when g is monotone [44]) and even for equations (1) (e.g. when g is a monotone and globally Lipschitzian function, with the Lipschitz constant $g'(0)$, and when additionally $K(s) = K(-s)$, $s \in \mathbb{R}$ [29, 41]). However, if the reaction term g is non-monotone and non-local, monotonicity and uniqueness are not longer obligatory front's characteristics. For example, [42] provides conditions sufficient for non-monotonicity of wavefronts' profiles for non-local equation (1) with compactly supported kernel K . Co-existence of multiple wavefronts for non-local models is also known from [21, 36]. Thus the question about the existence and uniqueness of *monotone* wavefronts for the monostable *non-monotone* non-local (or delayed) reaction-diffusion equations seems to be interesting and timely. In fact, recently it called the attention of several researchers. In this regard, the most studied model was the non-local KPP-Fisher equation [5, 7, 15, 21, 35, 36]

$$u_t(t, x) = u_{xx}(t, x) + u(t, x)(1 - \int_{\mathbb{R}} K(x - y)u(t, y)dy), \quad (3)$$

and its local delayed version [5, 13, 26, 22, 18, 19, 47] (called the diffusive Hutchinson's equation)

$$u_t(t, x) = u_{xx}(t, x) + u(t, x)(1 - u(t - \tau, x)). \quad (4)$$

The above cited papers elaborated a complete characterization of models (3) and (4) possessing monotone wavefronts. Moreover, the absolute uniqueness (i.e. uniqueness within the class of all wavefronts) of monotone wavefronts to (4) and the conditional uniqueness (i.e. uniqueness within the subclass of monotone wavefronts) of monotone wavefronts to (3) was also proved in these works. As we have mentioned, in general, monotone and non-monotone wavefronts can coexist in (3) [21, 36].

In the case of models (1) and (2) having non-monotone function g , the existence of *monotone* wavefronts was analyzed only for equation (2) in [19], by the help of the Hale-Lin functional-analytic approach and a continuation argument. This method required a detailed analysis of a family of linear differential Fredholm operators associated with (2). The discrete Lyapunov functionals of Mallet-Paret and Sell for delayed differential equations were also used in an essential way. Therefore the task of extension of the approach developed in [19] on non-local equations (1) seems to be quite difficult (if anyhow possible). Consequently, the main goal of the present paper is to provide an alternative technique allowing to analyse monotonicity of wavefronts for non-monotone and non-local equation (1). A key feature of this technique consists in reduction of the wave profile equation for (1) to a new and non-obvious convolution equation (see Section 2). The obtained nonlinear equation is then studied by means of various already established methods (in particular, we use the Berestycki-Nirenberg sliding solution method as well

as approaches developed in [2, 17]). Remarkably, by weakening restrictions imposed on the birth function g , our technique improves the existence criterion of [19]. Furthermore, it also enables us to prove the conditional uniqueness of monotone fronts for equations (1), (2). Note that for non-monotone equations (1), (2), the uniqueness problem was only partially solved in [2, 14, 19, 41] by means of the Diekmann-Kaper method [2, 12]. This, however, presumes the Lipschitz condition $|g(s_1) - g(s_2)| \leq g'(0)|s_1 - s_2|$ which is sufficient for the absolute uniqueness of each front (either monotone or non-monotone) but excludes from the consideration so-called pushed fronts. These fronts are quite significant from the ecological point of view, see [38, 44] and references therein. Our results (Theorems 4 and 6 below) provide a simple criterion guaranteeing the absolute uniqueness of monotone wavefronts for (2), including the minimal one (it doesn't matter whether it is pushed or not). It should be noted here that, as the results of [24, 19, 43] show, in many cases only the slow traveling fronts of (2) can be monotone (or eventually monotone) while the increase of propagation speed might lead to the appearance of slowly oscillating waves. This is why the studies of monotonicity of the slow traveling fronts to equations (1), (2) is of our primary interest. On the other hand, already starting from the seminal work of Kolmogorov, Petrovskii and Piskunov [25], the applied importance of the slowest (i.e. minimal or critical) waves is also well known: in particular, it reflects the fact that the invasion of a new unexplored territory by a single species population is realised with the so-called minimal speed of propagation c_* , in the form of the minimal front $u(x, t) = \phi_*(x + c_*t)$. Clearly, the geometric properties of the leading edge of the invasion profile $\phi_*(s)$, $s \in \mathbb{R}$, are quite significant for the description of transition from an uninvaded to invaded state.

In the subsequent part of this section, we state the key hypotheses used in the paper and briefly discuss our main theorems together with a key auxiliary assertion.

Main assumptions.

(M) $g \in C(\mathbb{R}_+)$, $g(s) > 0$ for $s > 0$, and the equation $g(s) = s$ has exactly two nonnegative solutions: 0 and $\kappa > 0$. Moreover, g is differentiable at the equilibria with $g'(0) > 1$ and $g'(\kappa) < 0$.

(ST) $g(s) - g'(\kappa)s$ is non-decreasing on $[0, \kappa]$. Observe that the last assumption implies the sub-tangency property of g at κ : $g(s) \leq g(\kappa) + g'(\kappa)(s - \kappa)$, $s \in [0, \kappa]$.

(K) $K \geq 0$ and $\int_{\mathbb{R}} K(s)ds = 1$. Moreover, $\int_{\mathbb{R}} K(s)e^{-\lambda s}ds < \infty$ for each $\lambda \in \mathbb{R}$.

Example 1. If g is differentiable on $[0, \kappa]$ then the hypothesis **(ST)** amounts to the inequality **(ST')**: $g'(s) \geq g'(\kappa)$ satisfied for all $s \in [0, \kappa]$. For the following non-local version of the popular Nicholson's blowflies diffusive equation

$$u_t(t, x) = u_{xx}(t, x) - \delta u(t, x) + p \int_{\mathbb{R}} K(x - y)u(t - h, y)e^{-u(t-h, y)}dy, \quad p > \delta > 0,$$

the assumptions **(M)** and **(ST')** are equivalent to the inequalities $e < p/\delta \leq e^2$. We note that the bulk of information concerning the Nicholson's diffusive equation is obtained for a simpler (monotone) case when $1 < p/\delta \leq e$ (cf. the recent works [48] and [39]).

Main results: existence. Clearly, $u(t, x) = \phi(x + ct)$ is a front solution of equation (1) if and only if the profile $y = \phi(t)$ solves the boundary value problem

$$y''(t) - cy'(t) - y(t) + \int_{\mathbb{R}} K(t-s)g(y(s-ch)) = 0, \quad y(-\infty) = 0, \quad y(+\infty) = \kappa, \quad y(t) \geq 0. \quad (5)$$

For kernel K satisfying **(K)**, we will consider the characteristic functions

$$\chi_0(z) = z^2 - cz - 1 + g'(0)e^{-chz} \int_{\mathbb{R}} e^{-zs} K(s) ds, \quad z \in \mathbb{C},$$

$$\chi_{\kappa}(z) = z^2 - cz - 1 + g'(\kappa)e^{-chz} \int_{\mathbb{R}} e^{-zs} K(s) ds, \quad z \in \mathbb{C},$$

associated with the linearizations of (5) at the equilibria 0 and κ , respectively. We will need the following three subsets $\mathcal{D}_0, \mathcal{D}_{\kappa}, \mathcal{D}_{\mathfrak{L}} := \overline{\mathcal{D}_0} \cap \mathcal{D}_{\kappa}$ of the half-plane $(h, c) \in \mathbb{R}_+ \times \mathbb{R}$:

$\mathcal{D}_{\kappa} = \{(h, c) \in \mathbb{R}_+ \times \mathbb{R} : \chi_{\kappa}(z) \text{ has at least one positive and one negative simple zeros}\};$

$\mathcal{D}_0 = \{(h, c) \in \mathbb{R}_+ \times \mathbb{R} : \chi_0(z) \text{ has exactly two positive zeros } \mu_0 < \mu_1\}.$

The geometric description of the open domain \mathcal{D}_0 is well known and it is summarised in the following assertion:

Proposition 2. *Assume that $g'(0) > 1$. Then for each $h \geq 0$ there exists a unique $c = c_{\#}(h) \in \mathbb{R}$ such that the equation $\chi_0(z) = 0$ with this c has a unique positive double root. The function $c_{\#} : \mathbb{R}_+ \rightarrow \mathbb{R}$ is C^∞ -continuous and strictly decreasing. Furthermore, \mathcal{D}_0 coincides with the set $\{(h, c) \in \mathbb{R}_+ \times \mathbb{R} : c > c_{\#}(h)\}$.*

PROOF. For example, see [17, Lemma 22] and [3, Theorem 1.1]. By [3] if, in addition, $K(s) = K(-s)$, $s \in \mathbb{R}$, then $0 < c_{\#}(h) = O(1/h)$ at $+\infty$. In general, however, $c_{\#}(h)$ can take negative values, cf. [17]. \square

It is clear that the finite part of the boundary of $\mathcal{D}_{\mathfrak{L}}$ consists from the curves determined either from the system $\chi_0(z) = 0$, $\chi'_0(z) = 0$ or from the system $\chi_{\kappa}(z) = 0$, $\chi'_{\kappa}(z) = 0$. Thus, in each particular case, the shape of the domains \mathcal{D}_{κ} , $\mathcal{D}_{\mathfrak{L}}$ can be easily identified. For instance, if $K(s)$ is the Dirac's delta, then $\mathcal{D}_{\mathfrak{L}}$ is a simply connected domain whose boundary contains a non-empty segment of the half-line $\{h = 0, c \geq 0\}$ [19]:

$$\mathcal{D}_{\mathfrak{L}} = \{(h, c) : h \in [0, h_*], \quad h_* \leq +\infty, \quad 0 < c_{\#}(h) \leq c < c^*(h)\},$$

where the smooth decreasing function $c^*(h)$ is determined by $\chi_{\kappa}(z)$. The aforementioned characteristics of $\mathcal{D}_{\mathfrak{L}}$ are essential for the use of a continuation argument in [19]. For general kernels K , however, the set $\mathcal{D}_{\mathfrak{L}}$ eventually might be more complicated (for instance, not connected). One of advantages of our present approach is that it does not require any connectedness property from $\mathcal{D}_{\mathfrak{L}}$:

Theorem 3. *Assume **(M)**, **(K)**, **(ST)** and that g is sub-tangential at the equilibrium 0: $g(s) \leq g'(0)s$, for all $s \in [0, \kappa]$. Then for each point (h, c) in the closure $\overline{\mathcal{D}_{\mathfrak{L}}}$ of the set $\mathcal{D}_{\mathfrak{L}}$, equation (1) has at least one wavefront $u(t, x) = \phi_c(x + ct)$ with strictly increasing profile $\phi_c(s)$, $s \in \mathbb{R}$.*

As we have mentioned, the conclusion and the proof of Theorem 3 are also valid when $K(s)$ is the Dirac delta function, i.e. for the local equation (2). Thus it is enlightening to compare criterion of front's monotonicity for (2) established in [19, Theorem 2.2] and Theorem 3. These two results almost coincide except for two important details: g in [19] must be more smooth ($C^{1,\gamma}$ -continuous on $[0, \kappa]$) and must have a unique critical point on $(0, \kappa)$. That is, the unimodal form of g is assumed in [19] instead of the condition (ST). Clearly, even if these both requirements are fulfilled for the classical population models (Nicholson's blowflies model, hematopoiesis model), they are independent: so that Theorems 3 and [19, Theorem 2.2] complement each other in the case of local delayed equations. This comparison also shows that for some models (e.g. for equation (2)) Theorem 3 provides the necessary and sufficient conditions of front's monotonicity. Indeed, it follows from [19] that (2) can not have any monotone front if (h, c) does not belong to the closure of the set $\mathcal{D}_{\mathcal{L}}$.

Main results: uniqueness. To prove the conditional uniqueness (up to a translation) of a monotone wavefront, instead of the above mentioned Diekmann-Kaper theory [2, 12, 48], here we are using an alternative approach based on the sliding solution method developed by Berestycki and Nirenberg, cf. [44]. This technique was successfully applied in [9, 10, 11, 30] to prove the uniqueness of *monotone* wavefronts without imposing any Lipschitz condition on g . In the present paper, we consider sliding solutions to prove the following.

Theorem 4. *Assume (M), (K) and (ST). In addition, let g be C^1 -smooth in some neighborhood of κ and there exist $C > 0$, $\theta \in (0, 1]$, $\delta > 0$ such that*

$$|g(u)/u - g'(0)| \leq Cu^\theta, \quad u \in (0, \delta]. \quad (6)$$

Fix some $(c, h) \in \mathcal{D}_{\mathcal{L}}$, and suppose that $u_1(t, x) = \phi(x + ct)$, $u_2(t, x) = \psi(x + ct)$ are two monotone traveling fronts of equation (1). Then $\phi(s) = \psi(s + s_0)$, $s \in \mathbb{R}$, for some s_0 .

Remark 5. *As a by-product of the proofs of Theorems 3, 4, we obtain the following: Assume (K) and (M) where $g'(\kappa) \geq 0$ is considered instead of $g'(\kappa) < 0$. If, in addition, $g(s) \leq g'(0)s$, $s \in [0, \kappa]$, and g is monotone and satisfies the smoothness conditions of Theorem 4, then for each point (h, c) in the closure of the set \mathcal{D}_0 , equation (1) has a unique (up to a translation) monotone wavefront $u(t, x) = \phi_c(x + ct)$.*

We will say that some c is an admissible speed of propagation for (1) (or for (2)) if there exists a positive wave solution $u = \phi(x + ct)$ to (1) (to (2), respectively) such that $\phi(-\infty) = 0$ and $\liminf_{t \rightarrow +\infty} \phi(t) > 0$. We call such a wave solution semi-wavefront. As [21, 36] reveals, proper semi-wavefronts and monotone fronts can co-exist in non-local monostable equations. Nevertheless, as the next result shows, the statement of Theorem 4 can be strengthened for the case of local delayed reaction:

Theorem 6. *Assume (M), (K) and (ST). Then each semi-wavefront $u = \phi(x + ct)$, $(h, c) \in \mathcal{D}_{\mathcal{L}}$, for equation (2) is actually a monotone front and therefore it is the unique possible wavefront solution of (2) (up to a translation) propagating with the speed c .*

Remark 7. *Fix some $h > 0$ and consider*

$$\mathcal{C}(h) := \{c \geq 0 : \text{there is a semi-wavefront for (2) propagating at the velocity } c\}.$$

It is well known (cf. [42, Theorem 4] or [24, Remark 8]) that $\mathcal{C}(h)$ contains some infinite subinterval $[c_1(h), +\infty)$ while $c_*(h) := \inf \mathcal{C}(h) \geq c_\#(h) \in \mathbb{R}$. It is easy to see that $\mathcal{C}(h)$ is closed (cf. [18, Lemma 26]) so that $c_*(h) \in \mathcal{C}(h)$. The number $c_*(h)$ is called the minimal speed of propagation for the monostable model (2). In general, $c_*(h)$ is not linearly determined, i.e. $\mathcal{C}(h) \neq [c_\#(h), +\infty)$ [23, 48]. The minimal wave is called pushed if $c_*(h) > c_\#(h)$. For equation (2) with monotone g , a min-max representation for the speed of pushed wavefront can be found in [38].

For some models (for example, if g is monotone on $[0, \kappa]$ [28, 44] or if g is sub-tangential at 0), we have $\mathcal{C}(h) = [c_*(h), +\infty)$. In addition, if g is also sub-tangential at κ , then $\mathcal{C}(h)$ for equation (2) can be represented as a union of three adjacent intervals $\mathcal{C}(h) = \mathcal{I}_m \cup \mathcal{I}_{so} \cup \mathcal{I}_{sw}$ (some of them can be empty) corresponding to velocities of monotone fronts, slowly oscillating fronts and proper semi-wavefronts, respectively [19, 43]. If g is not sub-tangential at κ , then also non-monotone non-oscillating fronts can appear [24]. In the important case, when g is neither monotone on $[0, \kappa]$ nor sub-tangential at 0, the question about the connectedness of the set $\mathcal{C}(h)$ is largely open at this point of investigation.

An auxiliary result. It is convenient to transform the profile equation (5) into a suitable nonlinear convolution equation [12, 47]

$$\phi(t) = \int_{\mathbb{R}} N(t-s)g_1(\phi(s-ch))ds, \quad (7)$$

with appropriate kernel $N \in L_1(\mathbb{R}, \mathbb{R}_+)$ and continuous monostable nonlinearity $g_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Certainly, N, g_1 depend on c, h, K, g and the choice of specific N, g_1 depends on the goals of investigation. A correct determination of N and g_1 may indicate a shortest way in establishing various properties of profiles (including their existence). For instance, all above mentioned front's monotonicity criteria for the KPP-Fisher equations (3) and (4) were obtained after discovering a satisfactory form of the associated convolution equation, see [15, 18, 26]. Similarly, an important part of this paper is focused on reducing equation (5) to the 'optimal' convolution equation:

Theorem 8. Assume **(M)** and **(K)**. Then for each point $(h, c) \in \mathcal{D}_{\mathfrak{L}}$, there exist g_1 , positive δ and kernels $N, -v > 0$ given by

$$N = -(1 + \xi)K * v := -(1 + \xi) \int_{\mathbb{R}} K(s)v(t-s)ds, \quad g_1(s) = \frac{g(s) + \xi s}{1 + \xi}, \quad \xi := |g'(\kappa)| + \delta,$$

such that the boundary value problem (5) has a solution if and only if equation (7) has a non-negative solution satisfying the boundary conditions of (5). Furthermore, $\int_{\mathbb{R}} N(s)ds = 1$ and $\int_{\mathbb{R}} N(s)e^{-\lambda s}ds < \infty$ for all λ from some maximal finite interval $(\gamma_l, \gamma_r) \ni \{0\}$. Continuous function v is C^∞ -smooth on \mathbb{R}_- and \mathbb{R}_+ and has a unique minimum point at $t = 0$. In fact, v is strictly monotone on \mathbb{R}_- and \mathbb{R}_+ and it is strictly convex on \mathbb{R}_- .

Remark 9. For equation (2), a more explicit form of $N(t)$ can be obtained:

$$N(t) = -(1 + \xi)v(t, \xi), \quad \text{where } v(t, \xi) = -\frac{1}{\chi'(\lambda_0(\xi))} \begin{cases} \tilde{u}(t), & t \geq 0, \\ e^{\lambda_0(\xi)t}, & t < 0, \end{cases}$$

$\chi(z) = z^2 - cz - 1 - \xi e^{-chz}$, $\lambda_0(\xi)$ is the unique positive zero of $\chi(z)$, and $\tilde{u}(t)$ is the solution of the following initial value problem:

$$u''(t) - cu'(t) - u(t) - \xi u(t - ch) = 0, \quad (8)$$

$$u(s) = e^{\lambda_0(\xi)s}, \quad s \in [-ch, 0], \quad u'(0) = -(\lambda_0(\xi) - c + \xi c h e^{-\lambda_0(\xi)ch}).$$

When $\xi = 0$, this formula for the fundamental solution $v(t, \xi)$ for (8) is well known, see (15). It should be noted that the explicit exponential form of $v(t)$ for negative t will allow to prove the monotonicity of all wavefronts under conditions of Theorem 6. On the other hand, the one-sided Laplace transform $\hat{v}(z) = \int_0^{+\infty} e^{-zt} v(t) dt$ of $v(t)$ can also be easily found:

$$\hat{v}(z) = \frac{1}{\chi(z)} - \frac{1}{\chi'(\lambda_0)} \frac{1}{z - \lambda_0(\xi)}.$$

This function is analytic in the half-plane $\{\Re z > \lambda_1(\xi)\}$ where $\lambda_1(\xi)$ is the biggest negative zero of $\chi(z)$. As the Laplace transform of the negative function, $-\hat{v}(x)$, $x \in (\lambda_1(\xi), +\infty)$, provides a new example of completely monotone elementary function, see [4, 31, 46].

Since the analytical properties of functions g_1, v defined in Theorem 8 are rather nice (for example, g_1 is monotone if **(ST)** is assumed), the optimality of their choice for solving our existence/uniqueness problems has to be explained in terms of optimality of the set $\mathcal{D}_{\mathfrak{L}} = \overline{\mathcal{D}_0} \cap \mathcal{D}_{\kappa}$. In the ideal case, the closure of $\mathcal{D}_{\mathfrak{L}}$ must contain the set \mathfrak{M} of all pairs (h, c) for which (5) has a unique (up to a shift) monotone solution. Since it is well known (cf. [17]) that $\mathfrak{M} \subseteq \overline{\mathcal{D}_0}$, we actually need only to justify the choice of

$$\mathcal{D}_{\kappa} = \{(h, c) \in \mathbb{R}_+ \times \mathbb{R} : \chi_{\kappa}(z) \text{ has at least one positive and one negative simple root}\}.$$

The necessity of the presence of at least one negative zero of $\chi_{\kappa}(z)$ in the definition of \mathcal{D}_{κ} for the existence of monotone fronts was proved both for the delayed equation (2) (e.g. see [19]) and the non-local equation (1) (at least when K has compact support, cf. [42, Theorem 6]). However, the necessity of the presence of at least one positive zero of $\chi_{\kappa}(z)$ in the definition of \mathcal{D}_{κ} is not so obvious. Here we give the following two arguments in support of this presence: firstly, in some situations (e.g. for equation (2)), at least one positive zero of $\chi_{\kappa}(z)$ exists automatically for all parameters in $\overline{\mathcal{D}_0}$; secondly, a unique positive zero of $\chi_{\kappa}(z)$ plays an essential role in proof of the monotonicity criterion in [19] (more precisely, in proofs of surjectivity of associated Fredholm operators, see Proposition 3.2 and Lemma 3.3 in [19]).

Finally, a few words about the organisation of the paper. In the next section, we define and study properties of the fundamental solution $v(t, \xi)$. These studies are resumed in Theorem 8 which is formally proved in Step I of Sections 3. The convolution equation (7) is then used to prove Theorems 3, 4 in Sections 3 and 4, respectively. The proof of Theorem 6 is given in Section 5.

2. Negativity of the fundamental solution.

2.1. The fundamental solution: definitions and properties

Fix $c, d, \xi \in \mathbb{R}$, kernel $K(s)$ satisfying **(K)** and consider the linear integral-differential inhomogeneous equation

$$y''(t) - cy'(t) - dy(t) - \xi \int_{\mathbb{R}} K(t-s)y(s-ch)ds + f(t) = 0, \quad (9)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded continuous function and the characteristic function

$$\chi(z, \xi) = z^2 - cz - d - \xi e^{-chz} \int_{\mathbb{R}} e^{-zs} K(s)ds, \quad z \in \mathbb{C},$$

does not have zeros on the imaginary axis (in such a case, we will say that equation (9) is hyperbolic). Suppose, for a moment, that f is compactly supported and that, for this inhomogeneity, equation (9) has a solution $y : \mathbb{R} \rightarrow \mathbb{R}$ exponentially decaying, together with its first derivative $y'(t)$, at $\pm\infty$. Then, applying the bilateral Laplace transformation to (9), we find easily that this equation has a solution $y(t) = -v * f(s)$, which is the convolution of f with the bilateral Laplace inverse $v(t, \xi)$ of $1/\chi(\lambda, \xi)$. Since $y(t)$ is a bounded function, the formula $y(t) = -v * f(s)$ shows that the inverse Laplace transform should be applied to $1/\chi(\lambda, \xi)$ considered on the maximal vertical analyticity strip $\Pi(\lambda_l, \lambda_r) := \{z : \lambda_l < \Re z < \lambda_r\}$ that includes the imaginary axis (observe that $\lambda_l < 0 < \lambda_r$ since the imaginary axis does not contain any singular point of $1/\chi(\lambda, \xi)$). The function $v(\cdot, \xi) : \mathbb{R} \rightarrow \mathbb{C}$ is called the fundamental solution for equation (9). The above said and the inversion theorem imply that

$$v(t, \xi) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{iut} du}{u^2 + ciu + d + \xi e^{-iuch} \int_{\mathbb{R}} K(s)e^{-ius}ds}, \quad t \in \mathbb{R}. \quad (10)$$

We view this formula as a formal definition of the fundamental solution for equation (9).

Lemma 10. *Suppose that $\chi(z, \xi)$ does not have pure imaginary zeros. Then $v(\cdot, \xi) : \mathbb{R} \rightarrow \mathbb{R}$ is a real valued function which is infinitely differentiable with respect to t on the set $\mathbb{R} \setminus \{0\}$ where it also satisfies equation (9) with $f(t) \equiv 0$. Next, there exist the limits $v'(0^-, \xi)$, $v'(0^+, \xi)$ and $v'(0^+, \xi) - v'(0^-, \xi) = 1$ (thus the limits $v''(0^-, \xi)$, $v''(0^+, \xi)$ exist and $v''(0^+, \xi) - v''(0^-, \xi) = c$).*

PROOF. Indeed, v is a real valued function because of the presentation

$$v(t, \xi) = -\frac{1}{\pi} \int_0^{+\infty} \frac{p(u) \cos(tu) + q(u) \sin(tu)}{p^2(u) + q^2(u)} du, \quad t \in \mathbb{R}, \quad (11)$$

where p, q satisfying $p(u) = p(-u)$, $q(u) = -q(-u)$, $u \in \mathbb{R}$, are defined by

$$p(u) := u^2 + d + \xi C(u), \quad q(u) := cu - \xi S(u),$$

where, due to the Lebesgue-Riemann lemma, the functions

$$C(u) := \int_{\mathbb{R}} K(s) \cos(u(ch+s))ds, \quad S(u) := \int_{\mathbb{R}} K(s) \sin(u(ch+s))ds,$$

are vanishing, together with their derivatives of all orders, at ∞ . In particular, this implies that

$$P(+\infty) = 0, \quad \text{where } P(u) := \frac{up(u)}{p^2(u) + q^2(u)},$$

while derivatives of all orders $k = 1, 2, 3, \dots$,

$$P^{(k)}(u) = (u^{-1})^{(k)}(1 + o(1)) = (-1)^k k! u^{-k-1}(1 + o(1)), \quad u \rightarrow +\infty,$$

are monotone at $+\infty$. Therefore, by the Dirichlet test of the uniform convergence of improper integrals [51, p. 421], the integral

$$\frac{1}{\pi} \int_0^{+\infty} \frac{up(u) \sin(tu)}{p^2(u) + q^2(u)} du$$

converges uniformly on each compact subset of $\mathbb{R} \setminus \{0\}$. In consequence, (cf. [51, p. 426]),

$$v'(t, \xi) = \frac{1}{\pi} \int_0^{+\infty} \frac{up(u) \sin(tu) - uq(u) \cos(tu)}{p^2(u) + q^2(u)} du, \quad t \neq 0, \quad (12)$$

exists for all $t \neq 0$. Note that the term $uq(u)/(p^2(u) + q^2(u))$ is Lebesgue integrable on \mathbb{R}_+ so that the function

$$I_2(t) = \int_0^{+\infty} \frac{uq(u) \cos(tu)}{p^2(u) + q^2(u)} du$$

is continuous on \mathbb{R} . Hence, in order to prove the existence of $v'(0^+, \xi)$, $v'(0^-, \xi)$, we only need to take into account the integral

$$I_1(t) := \int_0^{+\infty} P_0(u) \frac{\sin(tu)}{u} du = \int_0^{+\infty} (1 - P_1(u)) \frac{\sin(tu)}{u} du = \frac{\pi \operatorname{sign} t}{2} - \int_0^{+\infty} P_1(u) \frac{\sin(tu)}{u} du$$

where $t \neq 0$, $|u^{-1} \sin(tu)| \leq |t|$, $u > 0$, and

$$P_0(u) := \frac{u^2 p(u)}{p^2(u) + q^2(u)}, \quad P_1(u) = \frac{p(u)(1 + \xi C(u)) + q^2(u)}{p^2(u) + q^2(u)} \in L_1(\mathbb{R}_+).$$

Thus $I_1(0^+) = \pi/2$, $I_1(0^-) = -\pi/2$, so that $v'(0^+, \xi) - v'(0^-, \xi) = 1$.

Finally, in view of formulas (11), (12), a direct computation gives, for $t \neq 0$,

$$\begin{aligned} v'(t, \xi) - cv(t, \xi) - \int_0^t v(s, \xi) ds - \xi \int_0^t du \int_{\mathbb{R}} K(s) v(u - s - ch, \xi) ds = \\ \frac{1}{\pi} \int_0^{+\infty} \frac{(p^2(u) + q^2(u)) \sin(tu) + (q(u)p(u) - p(u)q(u)) \cos(tu)}{u(p^2(u) + q^2(u))} du + \\ \frac{1}{\pi} \int_0^{+\infty} \frac{\xi p(u)S(u) + \xi q(u)C(u) + dq(u)}{u(p^2(u) + q^2(u))} du = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \frac{\xi p(u)S(u) + \xi q(u)C(u) + dq(u)}{u(p^2(u) + q^2(u))} du. \end{aligned}$$

Consequently, $v''(t, \xi)$ exists for $t \neq 0$ and $v(t, \xi)$ satisfies equation (9) with $f(t) \equiv 0$ for all $t \neq 0$. In fact, if $t > 0$ (the case $t < 0$ is similar) then

$$v'(t, \xi) = \frac{1}{t\pi} \int_0^{+\infty} \left[P\left(\frac{v}{t}\right) \sin v + Q\left(\frac{v}{t}\right) \cos v \right] dv, \quad \text{where } Q(u) := \frac{-uq(u)}{p^2(u) + q^2(u)}.$$

This shows that all derivatives $v^{(j)}(t, \xi)$, $t > 0$, exist. □

It follows from (10) that $v(\pm\infty, \xi) = 0$ as the Fourier transform of a function from $L_1(\mathbb{R})$. In fact, some additional work shows that actually $v(t)$ is exponentially decaying at $\pm\infty$:

Lemma 11. *If equation (9) is hyperbolic then $v \in W^{1,1}(\mathbb{R})$. In addition, $|v(t)| \leq Ce^{-\gamma|t|}$, $t \in \mathbb{R}$, for some positive C, γ and $v'' \in L_1(\mathbb{R}_\pm)$ (so that $v'(\pm\infty, \xi) = 0$).*

PROOF. A simple inspection of the characteristic equation

$$z^2 - cz - d = \xi e^{-chz} \int_{\mathbb{R}} e^{-zs} K(s) ds, \quad (13)$$

shows that, in view of the hyperbolicity of (9), there exists $\gamma > 0$ such that the vertical strip $\{|\Re z| < 2\gamma\}$ does not contain roots of (13). But then we can shift the path of integration in the inversion formula for the Laplace transform (e.g. see [2, p. 88]) to obtain

$$v(t, \xi) = \frac{e^{\pm\gamma t}}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{iut} du}{(\pm\gamma + iu)^2 - c(\pm\gamma + iu) - d - \xi \int_{\mathbb{R}} K(s) e^{-(\pm\gamma + iu)s} ds} = e^{\pm\gamma t} \sigma_{\pm}(t),$$

where $\sigma_{\pm}(\infty) = 0$. Next, by (12),

$$v'(t, \xi) = -\frac{1}{2\pi} \lim_{T \rightarrow +\infty} \int_{-T}^T \frac{iue^{iut} du}{u^2 + ciu + d + \xi e^{-iuch} \int_{\mathbb{R}} K(s) e^{-ius} ds}, \quad t \neq 0.$$

Therefore similarly, for $t \neq 0$,

$$v'(t, \xi) = \frac{e^{\pm\gamma t}}{2\pi} \lim_{T \rightarrow +\infty} \int_{-T}^T \frac{(\pm\gamma + iu)e^{iut} du}{(\pm\gamma + iu)^2 - c(\pm\gamma + iu) + d - \xi \int_{\mathbb{R}} K(s) e^{-(\pm\gamma + iu)s} ds},$$

where the latter limit also exists in $L^2(\mathbb{R})$ and represent the Fourier transform of an element of $L_2(\mathbb{R})$. Thus $v'(t, \xi) = e^{\pm\gamma t} \rho_{\pm}(t)$, $t \neq 0$, where $\rho_{\pm} \in L_2(\mathbb{R})$. By the Hölder inequality, $v' \in L_1(\mathbb{R}_\pm)$ so that $v' \in L_1(\mathbb{R})$. Finally, since v satisfies the equation (9) on \mathbb{R}_\pm , we conclude that $v''(t) = cv'(t) + v(t) + \xi K * v(t - ch)$ also belongs to $L_1(\mathbb{R}_\pm)$. \square

Lemma 12. *If $v \in W^{1,1}(\mathbb{R})$ and function f is continuous and bounded then $v * f \in C_b^1(\mathbb{R})$ and $(v * f)' = v' * f$.*

PROOF. Clearly, $|v * f(t)| \leq \sup_{t \in \mathbb{R}} |f(t)| |v|_1$, $t \in \mathbb{R}$, and

$$v * f(t + \delta) - v * f(t) = \int_{\mathbb{R}} f(t - s)(v(s + \delta) - v(s)) ds = \int_{\mathbb{R}} f(t - s) \int_s^{s+\delta} v'(u) du ds =$$

$$\delta \int_{\mathbb{R}} v'(u) du \frac{1}{\delta} \int_{u-\delta}^u f(t - s) ds \quad \text{because of} \quad \int_{\mathbb{R}} \int_s^{s+\delta} |v'(u)| du ds = \delta |v|_1 < \infty.$$

$$\text{Since} \quad \left| \frac{1}{\delta} \int_{u-\delta}^u f(t - s) ds \right| \leq \sup_{t \in \mathbb{R}} |f(t)|, \quad \lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_{u-\delta}^u f(t - s) ds = f(t - u),$$

we conclude that $v * f$ is differentiable on \mathbb{R} and $(v * f)' = v' * f$. Note also that $|v' * f(t)| \leq \sup_{t \in \mathbb{R}} |f(t)| |v'|_1$, $t \in \mathbb{R}$. \square

Our next goal is, given a bounded continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ (i.e. $f \in C_b(\mathbb{R})$), to prove the uniqueness of bounded solution $y(t)$ for the hyperbolic equation (9) and to justify the representation $y(t) = -v * f(t)$:

Corollary 13. *Suppose that $f \in C_b(\mathbb{R})$. If equation (9) is hyperbolic and v is the associated fundamental solution then the formula $u = -v * f$ gives the unique C^2 -smooth bounded solution of (9).*

PROOF. Invoking Lemmas 10, 11, 12 and using the formula

$$u(t) = - \int_{-\infty}^t v(t-s, \xi) f(s) ds - \int_t^{+\infty} v(t-s, \xi) f(s) ds,$$

we find easily that

$$u'(t) = - \int_{\mathbb{R}} v'(t-s, \xi) f(s) ds, \quad u''(t) = - \int_{\mathbb{R}} v''(t-s, \xi) f(s) ds - f(t)(v'(0^+, \xi) - v'(0^-, \xi))$$

are continuous, bounded and satisfy equation (9).

On the other hand, assume that $u(t)$ is some classical bounded solution of equation (9). Then it is easy to see from (9) that $u'(t), u''(t)$ are also bounded on \mathbb{R} and

$$u'' * v(t) = \int_{\mathbb{R}} u''(s) v(t-s) ds = \int_{\mathbb{R}} v'(t-s) u'(s) ds = u(t) + \int_{\mathbb{R}} v''(t-s) u(s) ds,$$

$$u' * v(t) = \int_{\mathbb{R}} u'(s) v(t-s) ds = \int_{\mathbb{R}} u(s) v'(t-s) ds.$$

In consequence, considering the convolution of equation (9) with the fundamental solution, we find that $u(t) + v * f(t) = 0$. \square

2.2. Continuous function $v(t, \xi)$ as a distributional solution of a non-local equation

It is worthwhile to analyse the fundamental solution $v(t, \xi)$ and some of its properties from the point of view of the theory of distributions. The distributions will be regarded in the standard way, as elements of the dual space $\mathcal{D}'(\mathbb{R})$ [40] (recall that the space $\mathcal{D}(\mathbb{R})$ of test functions consists of compactly supported smooth functions). This perspective is quite useful since it helps to cope with more general non-local delayed differential operators

$$\mathcal{L}\phi(t) = \phi^{(n)}(t) + a_{n-1}\phi^{(n-1)}(t) + \dots + a_1\phi'(t) + a_0\phi(t) + b_0 \int_{\mathbb{R}} K(t-s)\phi(s) ds + \sum_{j=1}^m b_j\phi(t-h_j).$$

We assume here that $\phi \in \mathcal{D}(\mathbb{R})$, $n \geq 2$, $a_i, b_j, h_j \in \mathbb{R}$ and that the operator \mathcal{L} is hyperbolic in the sense that the characteristic function $\chi(z, \mathcal{L})$ determined from $(\mathcal{L})(e^{zt}) = \chi(z, \mathcal{L})e^{zt}$ does not have zeros on the imaginary axis. Obviously, this form of \mathcal{L} includes the particular case of the operator defined by the left-hand side of equation (9).

Consider the formally adjoint operator \mathcal{L}^* defined by

$$\mathcal{L}^*\psi(t) = (-1)^n\psi^{(n)}(t) + (-1)^{n-1}a_{n-1}\psi^{(n-1)}(t) + \dots - a_1\psi'(t) + a_0\psi(t) +$$

$$b_0 \int_{\mathbb{R}} K(s-t)\psi(s)ds + \sum_{j=1}^m b_j \psi(t+h_j), \quad \psi \in \mathcal{D}(\mathbb{R}).$$

Clearly, for all $\phi, \psi \in \mathcal{D}(\mathbb{R})$, it holds that $\mathcal{L}\phi, \mathcal{L}^*\psi \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} \psi(t)\mathcal{L}\phi(t)dt = \int_{\mathbb{R}} \phi(t)\mathcal{L}^*\psi(t)dt.$$

We have the following

Lemma 14. *Suppose that \mathcal{L} is hyperbolic. Then function*

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{iut} du}{\chi(iu, \mathcal{L})}, \quad t \in \mathbb{R}, \quad (14)$$

is continuous, bounded and Lebesgue integrable on \mathbb{R} . Moreover, it is a distributional solution of the equation $\mathcal{L}v(t) = \delta(t)$, where $\delta(t)$ is the Dirac delta function:

$$\int_{\mathbb{R}} v(t)\mathcal{L}^*\phi(t)dt = \phi(0) \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

In consequence, $v(t)$ is C^n -smooth on $\mathbb{R} \setminus \{0\}$ and $\mathcal{L}v(t) = 0$ for all $t \neq 0$.

PROOF. From $1/\chi \in L^1(\mathbb{R})$ we infer that $v \in C(\mathbb{R})$, $v(\pm\infty) = 0$. In fact, repeating the argument given in the first lines of the proof of Lemma 11, we obtain that $|v(t)| \leq Ce^{-\gamma|t|}$, $t \in \mathbb{R}$, for some positive C, γ . Now, using the Fubini theorem and integrating by parts, we find that

$$\begin{aligned} \int_{\mathbb{R}} v(t)\mathcal{L}^*\phi(t)dt &= \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{L}^*\phi(t)dt \int_{\mathbb{R}} \frac{e^{iut} du}{\chi(iu, \mathcal{L})} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{du}{\chi(iu, \mathcal{L})} \int_{\mathbb{R}} e^{iut} \mathcal{L}^*\phi(t)dt = \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{du}{\chi(iu, \mathcal{L})} \int_{\mathbb{R}} \phi(t) \mathcal{L}e^{iut} dt = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{du}{\chi(iu, \mathcal{L})} \int_{\mathbb{R}} \phi(t) \chi(iu, \mathcal{L}) e^{iut} dt = \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} du \int_{\mathbb{R}} \phi(t) e^{iut} dt = \phi(0). \end{aligned}$$

In the last line, we are using the inversion formula for the Fourier transform [51].

Finally, we observe that, on each interval (α, β) disjoint with $\{0\}$, continuous $v(t)$ is a distributional solution of the following inhomogeneous linear ordinary equation with constant coefficients and continuous right-hand side $F(t)$:

$$v^{(n)}(t) + a_{n-1}v^{(n-1)}(t) + \dots + a_1v'(t) + a_0v(t) = F(t), \quad t \in (\alpha, \beta),$$

$$F(t) := -b_0 \int_{\mathbb{R}} K(t-s)v(s)ds - \sum_{j=1}^m b_j v(t-h_j).$$

It is well known [37] that, in such a case, $v(t)$ is also a classical solution on (α, β) of the latter equation. \square

Now, since for $f \in C_b(\mathbb{R})$ and continuous $v \in L^1(\mathbb{R})$

$$-\int_{\mathbb{R}} v * f(t) \mathcal{L}^* \phi(t) dt = \int_{\mathbb{R}} f(s) ds \int_{\mathbb{R}} v(t-s) \mathcal{L}^* \phi(t+s) dt = \int_{\mathbb{R}} f(s) \phi(s) ds,$$

we obtain the following

Corollary 15. *For each fixed $f \in C_b(\mathbb{R})$, the function $u = -v * f$ is a distributional solution of the equation $\mathcal{L}u(t) + f(t) = 0$. In consequence [37], $u(t)$ is also a bounded C^n -smooth classical solution on \mathbb{R} of this equation.*

2.3. A criterion of negativity of the fundamental solution $v(t, \xi)$

In this subsection, assuming the hyperbolicity of equation (9), we establish a criterion of negativity of its fundamental solution $v(t, \xi)$. It is well known (and it is straightforward to check) that $v(t, \xi) < 0$, $t \in \mathbb{R}$, in the local and non-delayed case when $\xi = 0$, $d > 0$ and

$$v(s, 0) = \min\{e^{\lambda_1(0)s}, e^{\lambda_0(0)s}\} / (\lambda_1(0) - \lambda_0(0)), \quad (15)$$

with $\lambda_1(0) < 0 < \lambda_0(0)$ being the roots of the characteristic equation $\lambda^2 - c\lambda - d = 0$. Suppose now that $K(s)$ satisfies **(K)** and $\xi \geq 0$, $d + \xi > 0$. Then it is easy to see that there exists a unique positive number ξ^* such that $\chi(z, \xi)$ has both positive and negative finite zeros if and only if $\xi \in [0, \xi^*]$. In fact, since $\chi^{(4)}(s, \xi) < 0$ for all $s \in \mathbb{R}$, function $\chi(s, \xi)$, $s \in \mathbb{R}$, for $\xi \in (0, \xi^*]$ can have at most four real zeros $\lambda_j(\xi)$, all of them being finite if

$$\int_{-\infty}^0 K(s) ds \int_0^{+\infty} K(s) ds \neq 0.$$

In such a case, we will order them as $\lambda_2(\xi) \leq \lambda_1(\xi) < 0 < \lambda_0(\xi) \leq \lambda_{-1}(\xi)$. If $\int_{-\infty}^0 K(s) ds = 0$ and $\xi \in [0, \xi^*]$, then there are exactly two negative and one positive finite roots $\lambda_2(\xi) \leq \lambda_1(\xi) < 0 < \lambda_0(\xi)$; by definition, we set $\lambda_{-1}(\xi) = +\infty$. A similar situation occurs when $\int_0^{+\infty} K(s) ds = 0$, $\xi \in [0, \xi^*]$, where it is convenient to set $\lambda_2(\xi) = -\infty$. Finally, we set $\lambda_2(0) = -\infty$, $\lambda_{-1}(0) = +\infty$. Observe that in either case the biggest negative root $\lambda_1(\xi)$ and the smallest positive root $\lambda_0(\xi)$ are finite numbers.

Lemma 16. *Suppose that $\xi \in [0, \xi^*]$, $c^2 + d > 0$. Then in the closed strip*

$$\overline{\Pi}(\lambda_2, \lambda_{-1}) := \{z : \lambda_2(\xi) \leq \Re z \leq \lambda_{-1}(\xi)\}$$

the function $\chi(z, \xi)$ does not have zeros different from $\lambda_j(\xi)$, $j = -1, 0, 1, 2$.

PROOF. First, note that only zero of $\chi(z, \xi)$ on the line $\Re z = \lambda_j(\xi)$, where $j \in \{-1, 0, 1, 2\}$, is $\lambda_j(\xi)$. Indeed, if z_j , $\Re z_j = \lambda_j(\xi)$, denotes another root of equation (13) then, using the factorization $z^2 - cz - d = (z - A)(z - B)$ with real A, B , we get the following contradiction:

$$|\lambda_j^2 - c\lambda_j - d| < |z_j - A||z_j - B| = |z_j^2 - cz_j - d| \leq \xi e^{-ch\lambda_j} \int_{\mathbb{R}} e^{-\lambda_j s} K(s) ds = |\lambda_j^2 - c\lambda_j - d|.$$

Next, the right hand side of (13) is uniformly bounded in each closed strip $\overline{\Pi}(a, b)$ by

$$\xi^* \int_{\mathbb{R}} (e^{-a(ch+s)} + e^{-b(ch+s)}) K(s) ds,$$

while $z^2 - cz - 1 \rightarrow \infty$ as $z \rightarrow \infty$. In consequence, there exists positive C which does not depend on ξ such that each zero z_k of $\chi(z, \xi)$, $\xi \in [0, \xi^*]$, in $\overline{\Pi}(\lambda_1(\xi), \lambda_0(\xi))$ satisfies $|\Im z_k| \leq C$. Since $\lambda_j(\xi)$, $j = 0, 1$, are continuous functions of ξ and $\overline{\Pi}(\lambda_1(0), \lambda_0(0))$ does not contain non-real roots of $\chi(z, 0)$, we find that either $\overline{\Pi}(\lambda_1(\xi), \lambda_0(\xi))$ contains only two zeros of $\chi(z, \xi)$ for all $\xi \in [0, \xi^*]$ or there exists $\xi_1 \in (0, \xi^*]$ and complex zero z_1 of $\chi(z, \xi)$ such that $\Re z_1 \in \{\lambda_1(\xi_1), \lambda_0(\xi_1)\}$. However, as we have just proved, the latter can not happen.

Finally, suppose that $\chi(z_0, \xi) = 0$, $\lambda_2(\xi) < x_0 := \Re z_0 < \lambda_1(\xi)$, (the case when $\lambda_0(\xi) < \Re z_0 < \lambda_{-1}(\xi)$ can be treated analogously). Then we get the following contradiction

$$|z_0^2 - cz_0 - d| \leq \xi e^{-chx_0} \int_{\mathbb{R}} e^{-x_0 s} K(s) ds < |x_0^2 - cx_0 - d| \leq |z_0 - A||z_0 - B| = |z_0^2 - cz_0 - d|.$$

This completes the proof of the lemma. \square

Lemma 17. *Suppose that $\xi \in [0, \xi^*]$, $d > 0$, $h \geq 0$. Then $v(t, \xi) < 0$ for all $t \in \mathbb{R}$, $\xi \in [0, \xi^*]$. Moreover, $v(t, \xi)$ is sign-changing on \mathbb{R} for each $\xi > \xi^*$ close to ξ^* .*

PROOF. Due to Lemma 16, equation (9) is hyperbolic and therefore the fundamental solution exists. The proof of its negativity is divided in several steps. Recall that if $\xi \in [0, \xi^*)$ then $\chi'(\lambda_0(\xi)) > 0$, $\chi'(\lambda_1(\xi)) < 0$ and $\lambda_j(\xi)$, $j = 0, 1$, are simple zeros of $\chi(z, \xi)$.

Claim I. For each non-negative $\xi_0 < \xi^*$ there exist real numbers ν_0, ν_1 , a neighbourhood $\mathcal{O} \ni \xi_0$ and positive constants K, L such that, for all $\xi \in \mathcal{O}$,

$$\lambda_2(\xi) + L < \nu_1 < \lambda_1(\xi) - L, \quad \lambda_0(\xi) + L < \nu_0 < \lambda_{-1}(\xi) - L, \quad (16)$$

$$v(t, \xi) = \rho_j(\xi) e^{\lambda_j(\xi)t} + r_j(t, \xi), \quad |r_j(t, \xi)| \leq K e^{\nu_j t}, \quad (-1)^{j+1} t \geq 0, \quad j = 0, 1, \quad (17)$$

where $\rho_j(\xi) = (-1)^{j+1} / \chi'(\lambda_j(\xi)) < 0$, $j = 0, 1$, depend continuously on ξ .

We prove this claim for $j = 1$, the other case being similar. Fix some $\xi_0 < \xi^*$ and $\nu_1 \in (\lambda_2(\xi_0), \lambda_1(\xi_0))$. Then we can choose a neighbourhood $\mathcal{O} \ni \xi_0$ and $L > 0$ sufficiently small to meet the condition (16) for all $\xi \in \mathcal{O}$. Next, after moving the integration path in the inversion formula (10) from $\Re z = 0$ to $\Re z = \nu_1$, we obtain that, for $t \geq 0$, $v(\xi, t) =$

$$\frac{e^{\lambda_1 t}}{\chi'(\lambda_1, \xi)} + \frac{1}{2\pi i} \int_{\nu_1 - i\infty}^{\nu_1 + i\infty} \frac{e^{tz} dz}{\chi(z, \xi)} = -\frac{e^{\lambda_1 t}}{|\chi'(\lambda_1, \xi)|} + \frac{e^{\nu_1 t}}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{ist} ds}{\chi(\nu_1 + is, \xi)} =: e_1(t) + e^{\nu_1 t} q(t),$$

where $q(\pm\infty) = 0$ and

$$|q(t)| \leq K = \sup_{\xi \in \mathcal{O}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{ds}{|\chi(\nu_1 + is, \xi)|}.$$

Claim I implies that exponentially decaying function $v(t, \xi)$, $\xi \in \mathcal{O}$, is negative at $\pm\infty$. In particular, there exists a leftmost point $T_+(\xi) \geq 0$ such that $v(t, \xi) < 0$ for all $t > T_+(\xi)$. Analogously, $T_-(\xi) \leq 0$ denotes the rightmost point such that $v(t, \xi) < 0$ for all $t < T_-(\xi)$. By (15), $T_{\pm}(0) = 0$.

Claim II. $v(t, \xi)$ is a continuous function of $t \in \mathbb{R}$, $\xi \in [0, \xi^*]$. Furthermore, $v(t, \xi)$ is bounded on \mathbb{R} , uniformly with respect to $\xi \in [0, \xi^*]$.

Indeed, observe that the function $g(u, \xi) := 1/\chi(iu, \xi)$ is continuous on $\mathbb{R} \times [0, \xi^*]$ and $|g(u, \xi)| \leq G(u)$, $u \in \mathbb{R}$, $\xi \in [0, \xi^*]$, with $G \in L_1(\mathbb{R})$ defined by

$$G(u) = \begin{cases} \max\{1/|\chi(is, \xi)|, |s| \leq 2 + \xi^*, \xi \in [0, \xi^*]\}, & |u| \leq 2 + \xi^*, \\ 1/(u^2 - \xi^*), & |u| > 2 + \xi^*, \end{cases}$$

(recall that $\chi(\lambda, \xi) \neq 0$ when $\Re \lambda = 0$, $\xi \in [0, \xi^*]$). In particular, to prove the continuity statement, it suffices to apply the Lebesgue dominated convergence theorem to (10).

Claim III. On each closed interval $[0, \zeta] \subset [0, \xi^*)$, functions $T_{\pm}(\xi)$ are bounded.

Due to the compactness of $[0, \zeta]$, it is enough to prove that $T_{\pm}(\xi)$ are locally bounded. For example, consider $T_+(\xi)$ for $\xi \in \mathcal{O}$. We have that either $T_+(\xi) = 0$ or $T_+(\xi) > 0$ and

$$0 = v(T_+(\xi)) \leq \rho_1(\xi)e^{\lambda_1(\xi)T_+(\xi)} + Ke^{\nu_1 T_+(\xi)}, \quad \xi \in \mathcal{O}.$$

In the latter case, for all $\xi \in \mathcal{O}$,

$$T_+(\xi) \leq \frac{1}{\lambda_1(\xi) - \nu_1} \ln \frac{K}{|\rho_1(\xi)|} \leq \frac{1}{L} \ln(K \sup_{\xi \in \mathcal{O}} |\chi'(\lambda_1(\xi), \xi)|).$$

Claim IV. $v(t, \xi) < 0$ for all $t \in \mathbb{R}$, $\xi \in [0, \xi^*]$. Furthermore, $v(t, \xi)$ is sign-changing on \mathbb{R} for $\xi > \xi^*$ close to ξ^* .

Let $\xi_c \in [0, \xi^*]$ be the maximal number such that $v(t, \xi) < 0$, $t \in \mathbb{R}$, for all $\xi \in [0, \xi_c]$. For a moment, suppose that $\xi_c < \xi^*$. Since $v(t, 0) < 0$ for all $t \in \mathbb{R}$, due to Claims II and III, $\xi_c > 0$ and $v(t_c, \xi_c) = 0$ for some $T_-(\xi_c) \leq t_c \leq T_+(\xi_c)$. Since $v(\pm\infty, \xi_c) = 0$, there exists two numbers $a_c < b_c$ where $v(t, \xi_c)$ reaches its absolute minima on the half-lines $(-\infty, t_c]$ and $[t_c, +\infty)$, respectively. Clearly, either a_c or b_c is different from zero. For instance, suppose that $a_c \neq 0$. Then $v(t, \xi_c)$ is differentiable at this point where $v(a_c, \xi_c) < 0$, $v'(a_c, \xi_c) = 0$, $v''(a_c, \xi_c) \geq 0$ and $K * v(a_c - ch) \leq 0$. Obviously, this contradicts equation (9) with $f(t) \equiv 0$ at $t = a_c \neq 0$. This shows that $\xi_c = \xi^*$ and also implies that $v(t, \xi^*) \leq 0$ for all $t \in \mathbb{R}$. Now, $\xi = \xi^*$ is a bifurcation point for some real zero of $\chi(z, \xi)$: for instance, suppose that for $\lambda_2(\xi^*) = \lambda_1(\xi^*) < 0$. Clearly, $\chi'(\lambda_1(\xi^*), \xi^*) = 0$ while $\chi''(\lambda_1(\xi^*), \xi^*) \neq 0$ for otherwise $\lambda_1(\xi^*)$ would be a triple negative zero of $\chi(z, \xi^*)$. Then, using the inversion formula again, we find that $v(t, \xi^*)$ is negative at $+\infty$ because of the relation $\lim_{t \rightarrow +\infty} v(t, \xi^*)t^{-1}e^{-\lambda_1(\xi^*)t} = 1/\chi''(\lambda_1(\xi^*), \xi^*) < 0$. Thus the above proof of negativity also works for $v(t, \xi^*)$. Next, for all $\xi > \xi^*$ close to ξ^* , function $\chi(z, \xi)$ has two simple complex conjugated zeros $\lambda_{1\pm}(\xi) := p(\xi) \pm iq(\xi)$, $\lambda_{1\pm}(\xi^*) := \lambda_1(\xi^*)$, such that the strip $\Pi(p(\xi), 0]$ does not contain any zero of $\chi(z, \xi)$. Now, assuming that $v(t, \xi) \geq 0$, $t \in \mathbb{R}$, we infer from [46, Theorem 5b, p. 58] that the singularity of the Laplace transform $1/\chi(z, \xi)$ of $v(t, \xi)$, which is rightmost on the half-plane $\Re z \leq 0$, should be a real number and not complex as $\lambda_{1\pm}(\xi)$. This contradiction proves the second part of Claim IV. \square

Remark 18. It can be proved in Lemma 17 that actually $v(t, \xi)$ is oscillating (either at $+\infty$ or $-\infty$) for $\xi > \xi^*$ close to ξ^* . Observe that the assumption of smallness of $\xi - \xi^*$ is used only in order to assure the existence of zeros of $\chi(z, \xi)$ in the both half-planes $\{\Re z > 0\}$ and $\{\Re z < 0\}$. If K has a compact support (for example, if K is the delta of Dirac), this requirement is fulfilled automatically. However, we do not know whether the smallness condition can be omitted for general kernels K . This is because there is some

lack of a priori knowledge regarding the distribution of zeros of $\chi(z, \xi)$: nevertheless, since the entire function $\chi(iz, \xi)$ is of class A and is of completely regular growth, some general information about this distribution can be found in [27, Chapter V, Theorem 11].

Remark 19. The positivity of the fundamental solution (or of the Green function) for solving initial/boundary value problems for delayed differential equations is an important topic of the theory of functional differential equations. See the recent monographs [1, 16] for more references concerning this problem.

The final result of this section shows that the geometric form of $v(t, \xi)$ for $\xi \in [0, \xi^*]$ is quite similar to the shape of $v(t, 0)$ given in (15):

Corollary 20. If $\xi \in [0, \xi^*]$ then $v(t, \xi)$ has a unique minimum point at $t = 0$. Moreover, $v(t, \xi)$ is strictly monotone on \mathbb{R}_- and \mathbb{R}_+ . It is also strictly convex on \mathbb{R}_- .

PROOF. Indeed, as we have seen in the proof of Lemma 17, $v'(t, \xi)$ can not change the sign on $(-\infty, 0)$ and $(0, +\infty)$ because otherwise $v(t, \xi)$ reaches a local minimum at some point of $\mathbb{R} \setminus \{0\}$. By the same reason, $v'(t, \xi)$ can not vanish on an open interval. Finally, observe that all this implies that $v''(t, \xi) > 0$ for $t < 0$. \square

3. Proof of Theorem 3

Case I: $(h, c) \in \mathcal{D}_{\mathcal{L}}$. We are assuming that the characteristic equation $\chi_{\kappa}(z) = 0$ has at least one negative and one positive simple roots $\lambda_1(|g'(\kappa)|) < 0 < \lambda_0(|g'(\kappa)|)$. Therefore, for sufficiently small $\delta > 0$ we have that $\xi = |g'(\kappa)| + \delta \leq \xi^*$ and the equation $\chi(z, \xi) = 0$ also has at least one negative and one positive simple roots $\lambda_1(\xi), \lambda_0(\xi)$:

$$\lambda_1(\xi) < \lambda_1(|g'(\kappa)|) < 0 < \mu_0 < \mu_1 < \lambda_0(|g'(\kappa)|) < \lambda_0(\xi).$$

With $g_1(s) = (g(s) + \xi s)/(1 + \xi)$, the profile equation (5) can be rewritten as

$$y''(t) - cy'(t) - y(t) - \xi \int_{\mathbb{R}} K(t-s)y(s-ch)ds + (1+\xi) \int_{\mathbb{R}} K(t-s)g_1(y(s-ch)) = 0. \quad (18)$$

By Corollary 13, this equation has at least one bounded solution $\phi(t)$ if and only if

$$\phi(t) = N\phi(t), \text{ where } N\phi(t) := \int_{\mathbb{R}} N(t-s)g_1(\phi(s-ch))ds, \quad N(s) = -(1+\xi)v * K(s). \quad (19)$$

In virtue of Lemma 17, the following properties of $N(s)$ are immediate: $N(s) > 0$, $s \in \mathbb{R}$, $\int_{\mathbb{R}} N(s)ds = 1$, and

$$\int_{\mathbb{R}} e^{-zs}N(s)ds = -(1+\xi) \frac{\int_{\mathbb{R}} e^{-zs}K(s)ds}{\chi(z, \xi)} < \infty \text{ for all } z \in (\lambda_1(\xi), \lambda_0(\xi)). \quad (20)$$

On the other hand, $g_1(s)$ is strictly increasing on $[0, \kappa]$ where $g_1(\kappa) = \kappa$, $g_1(0) = 0$ and

$$g'_1(\kappa) = \frac{\delta}{|g'(\kappa)| + \delta} \in (0, 1), \quad g_1(s) = \frac{g(s) + \xi s}{1 + \xi} \leq g'_1(0)s = \frac{g'(0)s + \xi s}{1 + \xi}.$$

Therefore nonlinear convolution equation (19) can be analysed within the framework of theory developed in [17]. Particularly, Theorem 7 in [17] guarantees the existence of a positive solution $y = \phi(t)$ to (18) satisfying the conditions $\phi(-\infty) = 0$, $\phi(+\infty) = \kappa$. Moreover, it is easy to see that solution $\phi(t)$ provided by [17, Theorem 7] is a non-decreasing one if $g_1(s)$ is a non-decreasing function. For the sake of completeness, in Remark 21 below, we indicate the corresponding change in the proof of [17, Theorem 7]. Now, due to the positivity of $N(s)$, the profile $\phi(t)$ is actually a strictly increasing function: if $t_2 > t_1$ then $\phi(t_2 - s) \geq \phi(t_1 - s)$, $s \in \mathbb{R}$, $\phi(t_2 - s) \not\equiv \phi(t_1 - s)$, so that

$$\phi(t_2) = \int_{\mathbb{R}} N(s - ch)g_1(\phi(t_2 - s))ds > \int_{\mathbb{R}} N(s - ch)g_1(\phi(t_1 - s))ds = \phi(t_1).$$

Hence, the proof of Case I is completed if $g_1(s)$ is increasing on \mathbb{R}_+ . Otherwise, consider some increasing continuous and bounded function $g_2(s)$ coinciding with $g_1(s)$ on $[0, \kappa]$ and such that $g'_2(\kappa) = g'_1(\kappa)$. But then, due to the first part of the proof, convolution equation (19) where g_1 is replaced with g_2 has a monotone solution $\phi : \mathbb{R} \rightarrow [0, \kappa]$. Since $g_1(s) \equiv g_2(s)$ on $[0, \kappa]$, the same function $\phi(t)$ solves (19).

Case II: (h, c) belongs to the boundary of the set $\mathcal{D}_{\mathcal{L}}$. In such a case, there exists a sequence $\{(h_j, c_j)\}$ of points in $\mathcal{D}_{\mathcal{L}}$ converging to (h, c) . From Case I we conclude that for each point $\{(h_j, c_j)\}$ there exists a monotone positive solution $y = \phi_j(t)$, $\phi_j(-\infty) = 0$, $\phi_j(+\infty) = \kappa$, satisfying the profile equation

$$y''(t) - c_j y'(t) - y(t) + \int_{\mathbb{R}} K(t - s)g(y(s - c_j h_j)) = 0.$$

Since this equation is translation invariant, we can assume that $\phi_j(0) = \kappa/2$ for each j . Then it follows that $\phi_j(s)$ has a subsequence $\phi_{j_k}(t)$ converging (uniformly on compact subsets of \mathbb{R}) to a positive monotone solution $\phi(t)$, $\phi(0) = \kappa/2$, of the limit equation (5) (e.g. see [17] or [42, Section 6] for more details. Now, the monotonicity of $\phi(t)$ implies that the boundary conditions in (5) are also satisfied (e.g. see Remark 21 below). This completes the proof of Theorem 3. \square

Remark 21. *In order to solve the following slightly modified version*

$$\phi := \mathcal{A}\phi, \quad \text{where } \mathcal{A}\phi(t) := \int_{\mathbb{R}} N(s)\gamma_n(\phi(t - s - ch))ds, \quad (21)$$

$$\gamma_n(s) := \begin{cases} g'_1(0)s, & \text{for } s \in [0, 1/n], \\ \max\{g'_1(0)/n, g_1(s)\}, & \text{when } s \geq 1/n, \end{cases}$$

of equation $\mathcal{N}\phi = \phi$ in Section 4 of [17], we can use the iteration procedure $\phi_{j+1} = \mathcal{A}\phi_j$, $j = 0, 1, \dots$, $\phi_0(s) = n^{-1} \exp(\mu_0 s)$, $s \in \mathbb{R}$, instead of the Schauder fixed point theorem. For a small positive $\epsilon > 0$, set $\phi^-(t) = \phi_0(t)(1 - e^{\epsilon t})\chi_{\mathbb{R}_-}(t)$, where $\chi_{\mathbb{R}_-}(t)$ is the characteristic function of \mathbb{R}_- . Since $\gamma_n(s), \phi_0(t)$ are non-decreasing functions and $\phi_-(t) \leq \mathcal{A}\phi_-(t) \leq \phi_1(t) \leq \mathcal{A}\phi_0(t) \leq \phi_0(t)$, we conclude that each $\phi_j(t)$, $j \in \mathbb{N}$, is also a non-decreasing function and $\phi_0(t) \geq \phi_2(t) \geq \dots \geq \phi_j(t) \geq \dots \geq \phi_-(t)$. Then the limit $\phi(t) = \lim_{j \rightarrow +\infty} \phi_j(t)$ should be a positive non-decreasing and bounded solution of the equation $\phi = \mathcal{A}\phi$. Taking the limit in (21) as $t \rightarrow \pm\infty$, we obtain that $\phi(\pm\infty) = \gamma_n(\phi(\pm\infty))$ that immediately implies that $\phi(-\infty) = 0$, $\phi(+\infty) = \kappa$.

4. Proof of Theorem 4

In this section, we show how the use of convolution equation (18) helps to extend the front uniqueness result established for equation (2) with monotone birth function g (e.g. see [44, Theorem 1.2]) on the case of non-local and non-monotone model (1).

Lemma 22. *Fix some $(c, h) \in \mathcal{D}_{\mathcal{L}}$ and suppose that $\phi, \psi : \mathbb{R} \rightarrow (0, \kappa]$ are two wavefront profiles satisfying equation (5) and such that ϕ is monotone and, for some finite T ,*

$$\phi(t) < \psi(t), \quad t < T. \quad (22)$$

Then $\phi(t) < \psi(t)$ for all $t \in \mathbb{R}$.

PROOF. Set $a_* = \inf \mathcal{A}$ where

$$\mathcal{A} := \{a \geq 0 : \psi(t) + a \geq \phi(t), \quad t \in \mathbb{R}\}.$$

Note that $\mathcal{A} \neq \emptyset$ since $[\kappa, +\infty) \subset \mathcal{A}$. Clearly, $a_* \in \mathcal{A}$.

Now, if $a_* = 0$ then $\psi(t) \geq \phi(t)$, $t \in \mathbb{R}$. We claim that, in fact, $\psi(t) > \phi(t)$, $t \in \mathbb{R}$. Indeed, otherwise we can suppose that T is such that $\phi(T) = \psi(T)$. In this way, the difference $\psi(t) - \phi(t) \geq 0$ reaches its minimal value 0 at T . Then, recalling that $N(t) = -(1 + \xi)v * K(t) > 0$ for all $t \in \mathbb{R}$, we get a contradiction:

$$0 = \psi(T) - \phi(T) = \int_{\mathbb{R}} N(s - ch)(g_1(\psi(T - s)) - g_1(\phi(T - s)))ds > 0. \quad (23)$$

In this way, Lemma 22 is proved when $a_* = 0$ and, consequently, we have to consider the case $a_* > 0$. Let $\sigma > 0$ be small enough to satisfy

$$\gamma_1 := \max_{s \in [\kappa - \sigma, \kappa + \sigma]} g'_1(s) < 1.$$

Case I. First, we take $Q > 0$ such that $\kappa \int_Q^{+\infty} N(s - ch)ds < a_*(1 - \gamma_1)$ and suppose that T is large enough to have

$$\phi(t), \psi(t) \in (\kappa - \sigma, \kappa + \sigma), \quad t \geq T - Q. \quad (24)$$

In such a case non-negative function

$$w(t) := \psi(t) + a_* - \phi(t), \quad w(\pm\infty) = a_* > 0,$$

reaches its minimal value 0 at some leftmost point t_m , where $\psi(t_m) - \phi(t_m) = -a_*$. Thus $\psi(t_m) < \phi(t_m)$ and therefore $t_m > T$ so that

$$\psi(t_m - s), \phi(t_m - s) \in (\kappa - \sigma, \kappa + \sigma), \quad s \leq Q.$$

In consequence, setting $\theta(s) \in (\kappa - \sigma, \kappa + \sigma)$, we obtain

$$\begin{aligned} -a_* &= \psi(t_m) - \phi(t_m) = \int_{\mathbb{R}} N(s - ch)(g_1(\psi(t_m - s)) - g_1(\phi(t_m - s)))ds = \\ &= \left(\int_{-\infty}^Q + \int_Q^{+\infty} \right) N(s - ch)(g_1(\psi(t_m - s)) - g_1(\phi(t_m - s)))ds > \\ &= -a_*(1 - \gamma_1) + \int_{-\infty}^Q N(s - ch)g'_1(\theta(s))(\psi(t_m - s) - \phi(t_m - s))ds \geq \\ &= -a_* + a_*\gamma_1 - \gamma_1 a_* \int_{-\infty}^Q N(s - ch)ds \geq -a_*, \quad \text{a contradiction.} \end{aligned}$$

Case II. If (24) does not hold, then, due to the convergence of profiles at $+\infty$ and the strict monotonicity of ϕ , we can find large $\tau > 0$ and $T_1 > T$ such that

$$\psi(t + \tau) > \phi(t), \quad t < T_1, \quad \phi(t), \psi(t + \tau) \in (\kappa - \sigma, \kappa + \sigma), \quad t \geq T_1 - Q.$$

Therefore, in view of the previous arguments, we obtain that

$$\psi(t + \tau) > \phi(t), \quad t \in \mathbb{R}. \quad (25)$$

Define now τ_* by

$$\tau_* := \inf\{\tau \geq 0 : \text{inequality (25) holds}\}.$$

It is clear that $\psi(t + \tau_*) \geq \phi(t)$, $t \in \mathbb{R}$. Now, using the same argument as in (23), we conclude that either $\psi(t + \tau_*) \equiv \phi(t)$ with $\tau_* = 0$ (a contradiction) or $\psi(t + \tau_*) > \phi(t)$, $t \in \mathbb{R}$. In the latter case, if $\tau_* = 0$, then Lemma 22 is proved. Otherwise, $\tau_* > 0$ and for each $\varepsilon \in (0, \tau_*)$ there exists a unique $T_\varepsilon > T$ such that

$$\psi(t + \tau_* - \varepsilon) > \phi(t), \quad t < T_\varepsilon, \quad \psi(T_\varepsilon + \tau_* - \varepsilon) = \phi(T_\varepsilon).$$

It is immediate to see that $\lim_{\varepsilon \rightarrow 0^+} T_\varepsilon = +\infty$. Indeed, if $T_{\varepsilon_j} \rightarrow T'$ for some finite T' and $\varepsilon_j \rightarrow 0^+$, then we get a contradiction: $\psi(T' + \tau_*) = \phi(T')$. Therefore, if ε is small, then

$$\psi(t + \tau_* - \varepsilon), \phi(t) \in (\kappa - \sigma, \kappa + \sigma), \quad t \geq T_\varepsilon - Q,$$

that is $\psi(t + \tau_* - \varepsilon)$ and $\phi(t)$ satisfy condition (24). But then we get $\psi(t + \tau_* - \varepsilon) > \phi(t)$ for all $t \in \mathbb{R}$, in contradiction to the definition of τ_* . This means that $\tau_* = 0$ and the proof of Lemma 22 is completed. \square

Corollary 23. *Fix some (c, h) in the closure of \mathcal{D}_0 and suppose that equation (5) possesses two monotone wavefronts ϕ and ψ . Then there exists $s_0, s_1 \in \mathbb{R}$ and $j \in \{0, 1\}$ such that either $\lim_{s \rightarrow -\infty} \phi(s + s_0)e^{-\mu_j s} = 1$, $\lim_{s \rightarrow -\infty} \psi(s + s_1)e^{-\mu_j s} = 1$, or $\lim_{s \rightarrow -\infty} \phi(s + s_0)e^{-\mu_j s} s^{-1} = -1$, $\lim_{s \rightarrow -\infty} \psi(s + s_1)e^{-\mu_j s} s^{-1} = -1$.*

PROOF. First, we prove that every profile satisfies one of the given asymptotic formula, with j which might depend on the profile. For definiteness, we will take profile ϕ . We are going to apply some results of [2] to the convolution equation (19). It follows from (20) that the set $\{z : \sigma_K < \Re z < \gamma_K\}$, where $\sigma_K = \lambda_1(\xi)$, $\gamma_K = \lambda_0(\xi)$, is the maximal open strip of convergence for the Laplace transform of N , cf. [46, Theorem 16b]. Moreover,

$$\lim_{x \rightarrow \gamma_K -} \int_{\mathbb{R}} N(s) e^{-sx} ds = +\infty \text{ and, in virtue of (17), } N(s) = O(e^{\lambda_0(\xi)s}), \quad s \rightarrow -\infty.$$

Therefore, using condition (6) and a standard argument of the Diekmann-Kaper approach (cf. Step I of the proof of Theorem 3 in [2]), we find that, for some $j, k \in \{0, 1\}$ and $\rho > 0$, the Laplace transform $\int_{\mathbb{R}} \phi(s) e^{-zs} ds$ is analytic in the strip $0 < \Re z < \mu_j$, has a singularity at μ_j , and satisfies

$$\frac{\chi_0(z)}{\chi(z, \xi)} \int_{\mathbb{R}} \phi(s) e^{-zs} ds = D(z),$$

where $D(z)$ is analytic in a bigger strip $0 < \Re z < \mu_j + \rho$. Since clearly $\Phi_+(z) := \int_0^{+\infty} \phi(s)e^{-zs}ds$ is analytic in the half-plane $\{\Re z > 0\}$, we conclude that the function $Q(z) := D(z)\chi(z, \xi)/\chi_0(z) - \Phi_+(z)$ is meromorphic in $0 < \Re z < \mu_j + \rho$, where it has a unique singularity (a simple or double pole) at μ_j . Since $\Phi_-(z) := \int_{-\infty}^0 e^{-sz}\phi(s)ds = Q(z)$ for $\Re z \in (0, \mu_j)$ and $\phi(s)$ is positive and non-decreasing on \mathbb{R}_- , an application of the Ikehara theorem [8, Proposition 2.3] yields the required asymptotic formula.

Finally, we claim that ϕ and ψ have the same asymptotic behavior at $-\infty$. For example, suppose that $\phi(t) \sim e^{\mu_0 t}$ and $\psi(t) \sim e^{\mu_1 t}$ as $t \rightarrow -\infty$. Then for every fixed $\tau \in \mathbb{R}$ there exists $T(\tau)$ such that $\psi(t + \tau) < \phi(t)$ for all $t < T(\tau)$. Applying Lemma 22, we obtain that $\psi(s) < \phi(t)$ for every $s := t + \tau$, $t \in \mathbb{R}$, what obviously is false. \square

Now we are in position to finalise the proof of Theorem 4. By Corollary 23, we can suppose that $\psi(t)$ and $\phi(t)$ have the same type of asymptotic behavior at $-\infty$. Consequently, $\psi(t + \tau), \phi(t)$ satisfy condition (22) of Lemma 22 for every small $\tau > 0$. But then $\psi(t + \tau) > \phi(t)$ for every small $\tau > 0$ that yields $\psi(t) \geq \phi(t)$, $t \in \mathbb{R}$. By symmetry, we also find that $\phi(t) \geq \psi(t)$, $t \in \mathbb{R}$, and Theorem 4 is proved. \square

5. Proof of Theorem 6

We will show that conditions of Theorem 6 assure that each semi-wavefront $u = \phi(x + ct)$, $(h, c) \in \mathcal{D}_{\mathfrak{L}}$, of equation (2) is actually a monotone wavefront. Indeed, it is easy to see that $0 < \phi(t) < \kappa$, $t \in \mathbb{R}$, since otherwise, without loss of generality, we can assume that $\phi(t_0) = \max_{s \in \mathbb{R}} \phi(s)$ for some t_0 that leads to the following contradiction:

$$\kappa \leq \phi(t_0) = \max_{s \in \mathbb{R}} \phi(s) = \int_{\mathbb{R}} N(t_0 - s)g_1(\phi(s - ch))ds < \int_{\mathbb{R}} N(t_0 - s) \max_{s \in \mathbb{R}} g_1(\phi(s))ds \leq \phi(t_0).$$

Next, we will need the following

Lemma 24. *Set $\Gamma(s) := g_1(\phi(s - ch))$. If the semi-wavefront $\phi(t)$ is increasing on \mathbb{R}_- and satisfies $\phi'(0) = 0$ then, for $t \in [0, ch]$,*

$$\phi'(t) = \int_{-\infty}^0 (N(t - s) - e^{\lambda_0(\xi)t}N(-s))d\Gamma(s) + \int_0^t (N(t - s) - e^{\lambda_0(\xi)(t-s)}N(0))d\Gamma(s).$$

PROOF. Since $\Gamma(s)$ increases on $(-\infty, ch]$ and $\Gamma(-\infty) = 0$, all Riemann-Stieltjes integrals in the above formula are well defined and convergent. Next, note that $g(s) = g_1(s)(1 + \xi) - \xi s$ is of bounded variation on $[0, \kappa]$. Thus, using [43, Remark 9(2)] together with Corollary 23, we conclude that $\phi(t)$ can have at most a finite number of critical points on each interval $(-\infty, \alpha]$. This implies that $\Gamma(s)$ has bounded variation on each $(-\infty, \alpha]$. Next, in view of Remark 9, after integrating by parts, we find that

$$\begin{aligned} \phi'(t) &= \int_{-\infty}^t N'(t - s)\Gamma(s)ds + \int_t^{+\infty} \lambda_0(\xi)N(0)e^{\lambda_0(\xi)(t-s)}\Gamma(s)ds = \\ &= \int_{-\infty}^t N(t - s)d\Gamma(s) + \int_t^{+\infty} N(0)e^{\lambda_0(\xi)(t-s)}d\Gamma(s) = \end{aligned}$$

$$\int_{-\infty}^t N(t-s)d\Gamma(s) + e^{\lambda_0(\xi)t}N(0) \left(\int_0^{+\infty} e^{-\lambda_0(\xi)s}d\Gamma(s) - \int_0^t e^{-\lambda_0(\xi)s}d\Gamma(s) \right).$$

Since $\phi'(0) = 0$, it holds that

$$N(0) \int_0^{+\infty} e^{-\lambda_0(\xi)s}d\Gamma(s) = - \int_{-\infty}^0 N(-s)d\Gamma(s)$$

and therefore

$$\begin{aligned} \phi'(t) &= \int_{-\infty}^t N(t-s)d\Gamma(s) + e^{\lambda_0(\xi)t} \left(- \int_{-\infty}^0 N(-s)d\Gamma(s) - N(0) \int_0^t e^{-\lambda_0(\xi)s}d\Gamma(s) \right) = \\ &= \int_{-\infty}^0 (N(t-s) - e^{\lambda_0(\xi)t}N(-s))d\Gamma(s) + \int_0^t (N(t-s) - N(0)e^{\lambda_0(\xi)(t-s)})d\Gamma(s). \quad \square \end{aligned}$$

Theorem 25. *Assume **(M)**, **(K)** and **(ST)**. Then each semi-wavefront $u = \phi(x + ct)$, $(h, c) \in \mathcal{D}_{\mathcal{L}}$, of equation (2) is a monotone wavefront.*

PROOF. From [43, Lemma 6], we know that $\phi'(t) > 0$ on some maximal interval $(-\infty, \sigma)$. If $\sigma = +\infty$, the corollary is proved. If σ is finite, without loss of generality we may take $\sigma = 0$. Then $\Gamma(t) = g_1(\phi(t - ch))$ is strictly increasing on $(-\infty, ch)$. But then Lemma 24 implies that $\phi'(t) \leq 0$ for all $t \in (0, ch]$. Here, we are using the inequalities

$$N(t-s) < N(-s) < e^{\lambda_0(\xi)t}N(-s), \quad s \leq 0 < t; \quad N(t-s) < N(0) < N(0)e^{\lambda_0(\xi)(t-s)}, \quad s < t.$$

Thus $\phi'(t) \leq 0$ on some maximal interval $(0, \sigma_1)$. Note that σ_1 must be a finite real number since otherwise $\phi'(t) \leq 0$ on $(0, +\infty)$ implying $\phi(+\infty) = 0$. However, this contradicts the uniform persistence property of semi-wavefronts [17]. In consequence, $\sigma_1 > ch$ is finite so that $\phi'(\sigma_1) = 0$, $\phi''(\sigma_1) \geq 0$ and $\phi(\sigma_1) \leq \phi(\sigma_1 - ch)$. On the other hand, we know that $\phi''(t) - c\phi'(t) - \phi(t) + g(\phi(t - ch)) = 0$ for all $t \in \mathbb{R}$ so that

$$\phi''(\sigma_1) - \phi(\sigma_1) + g(\phi(\sigma_1 - ch)) = 0,$$

from which we obtain $\kappa > \phi(\sigma_1 - h) \geq \phi(\sigma_1) \geq g(\phi(\sigma_1 - h)) > 0$, a contradiction. \square

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